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EQUATIONS OF THE LINEAR THEORY OF ELASTICITY OF ANISOTROPIC MATERIALS, REDUCED TO THREE INDEPENDENT WAVE EQUATIONS

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Characteristic operators and vectors for the system of differential equations of the linear theory of elasticity were introduced in [1-3]. Determination of the operators and vectors was reduced in [3] to a special coupled eigenvalue problem for six numerical matrices comprising the components of the elastic modulus tensor. Here, we propose a method of structuring the matrix of the operators of the linear theory of elasticity for anisotropic materials in such a way as to permit reduction of the initial system to three independent wave equations. We find specific classes of anisotropic materials that depend on arbitrary parameters. In particular, we obtain formulas to describe an elastic anisotropic medium (generalization of a Green medium) with longitudinal and transverse waves for any direction of the wave normal. Formulas for special orthotropic and transversely isotropic materials are also obtained.

With arbitrary anisotropy and an absence of body forces, the equations of the theory of elasticity appear as follows in orthotropic cartesian coordinates x_1, x_2, x_3 [4]

$$L_{ij}u_j = 0,\ L_{ij} = L_{ji} = A_{i(kl)j}\partial_{kl} - \rho\delta_{ij}\partial_{..}, \tag{1}$$

where u_j is the displacement vector; $A_{i(kl)j} = (A_{iklj} + A_{ilkj})/2$; $A_{iklj} = A_{kilj} = A_{ljik}$ is the constant tensor of the elastic moduli; ρ is the constant density of the material; δ_{ij} is the Kronecker symbol; $\partial_{.}$ and ∂_k are derivatives with respect to time and the coordinate x_k; repeating letter subscripts denote summation. The properties of the coefficients $A_{i(kl)j}$ were studied in [5-7].

For operators (1), we attempt to find differential matrices $T = [t_{jp}]$, $D = \mathrm{diag}(D_1, D_2, D_3)$ with constant coefficients such that $LT = TD$, $|T| \neq 0$. Then the general solution of Eqs. (1) will be as follows: $u = T\varphi$, $D\varphi = f$, $Tf = 0$. The formulas $u = T\varphi$, $\varphi = T'\tilde{u}$ (the prime denoting transposition) transpose the solutions of the equations $Lu = 0$, $D\varphi = 0$ [2, 3]. The expression $u = TT'\tilde{u}$ is the formula for obtaining new solutions, i.e. $Q = TT'$ is a symmetry operator in the sense of group analysis [8].

Let $D_1 = a_{kl}^{(1)}\partial_{kl} - \rho\partial_{..}$, $a_{kl}^{(1)} = a_{lk}^{(1)}$, $t_{jl} = \alpha_{js}\partial_s$ We then write the relation $LT = TD$ in the form

$$(A_{i(kl)j} - \delta_{ij}a_{kl}^{(1)})\alpha_{js}\partial_{kls} = 0. \tag{2}$$

Reducing similar terms in (2) and equating the coefficients with ∂_{kls} to zero, we can use (2) to obtain [3, 9] a special coupled eigenvalue problem for six matrices $A^{(1)} = A_{i(11)j}, \ldots, A^{(6)} = \sqrt{2A_{i(12)j}}$:

$$\begin{gathered}
(A_{i(11)j} - \delta_{ij}a_{11})\alpha_{j1} = 0,\ (A_{i(22)j} - \delta_{ij}a_{22})\alpha_{j2} = 0,\\
(A_{i(33)j} - \delta_{ij}a_{33})\alpha_{j3} = 0;\\
2(A_{i(23)j} - \delta_{ij}a_{23})\alpha_{j2} + (A_{i(22)j} - \delta_{ij}a_{22})\alpha_{j3} = 0,\\
(A_{i(33)j} - \delta_{ij}a_{33})\alpha_{j2} + 2(A_{i(23)j} - \delta_{ij}a_{23})\alpha_{j3} = 0;\\
2(A_{i(13)j} - \delta_{ij}a_{13})\alpha_{j1} + (A_{i(11)j} - \delta_{ij}a_{11})\alpha_{j3} = 0,\\
(A_{i(33)j} - \delta_{ij}a_{33})\alpha_{j1} + 2(A_{i(13)j} - \delta_{ij}a_{13})\alpha_{j3} = 0;
\end{gathered}$$

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$$2(A_{i(12)j} - \delta_{ij}a_{12})\alpha_{j1} + (A_{i(11)j} - \delta_{ij}a_{11})\alpha_{j2} = 0,$$
$$(A_{i(22)j} - \delta_{ij}a_{22})\alpha_{j1} + 2(A_{i(12)j} - \delta_{ij}a_{12})\alpha_{j2} = 0;$$
$$2[(A_{i(23)j} - \delta_{ij}a_{23})\alpha_{j1} + (A_{i(13)j} - \delta_{ij}a_{13})\alpha_{j2} + (A_{i(12)j} - \delta_{ij}a_{12})\alpha_{j3}] = 0. \tag{3}$$

Here, $a_{kl} = a_{kl}^{(1)}$. System (3) consists of 30 equations and is difficult to solve directly. However, if we assign α_{js}, then it becomes fairly simple to find the eigenvalues a_{kl} and the corresponding matrices $A^{(1)}, \ldots, A^{(6)}$ and, thus [6, 7], the matrices A_{ij} of the elastic moduli in the generalized Hooke's law.

The matrix T has the following structure [3]:

$$T = [\alpha_{js}\partial_s, \beta_{jp}\partial_p, \varepsilon_{jmn}\alpha_{ms}\beta_{np}\partial_{sp}] \tag{4}$$

(ε_{jmn} are Levi–Civita symbols), while

$$\alpha'\beta + \beta'\alpha = 0 \tag{5}$$

is the condition of orthogonality of the columns t_{j1}, t_{j2}. It follows from (5) that $\alpha'\beta = c$ is an antisymmetric matrix ($c' = -c$). If $|\alpha| \neq 0$, then $\beta = (\alpha')^{-1}c = (\alpha')^{-1}\alpha'\tilde{c}\alpha = \tilde{c}\alpha$, $\tilde{c}' = -\tilde{c}$. Matrix (4) can be written as

$$T = [t_{j1}, \varepsilon_{jmn}c_m t_{n1}, c_j t_{p1} t_{p1} - c_n t_{n1} t_{j1}], \tag{6}$$

where t_{j1} and c_m are arbitrary non-colinear vectors. The determinant of matrices (4) and (6) has the form

$$|T| = t_{j3}t_{j3} = (t_{i1}t_{i1})(t_{k2}t_{k2}) = (t_{i1}t_{i1})(c_m c_m t_{n1} t_{n1} - c_m t_{m1} c_{n1} t_{n1}).$$

Let a certain symmetric matrix A be represented in terms of eigenvalues and orthonormalized eigenvectors ($f_{ip}f_{iq} = \delta_{pq}$):

$$A = F\Lambda F' \to \Lambda = F'AF, \lambda_{pq} = A_{ij}f_{ip}f_{jq}.$$

We use this to obtain the following eigenvalues for $p = q$

$$\lambda_{11} = A_{ij}f_{i1}f_{j1}, \lambda_{22} = A_{ij}f_{i2}f_{j2}, \lambda_{33} = A_{ij}f_{i3}f_{j3}, \tag{7}$$

For $p \neq q$

$$\lambda_{21} = A_{ij}f_{i2}f_{j1} = 0, \lambda_{31} = A_{ij}f_{i3}f_{j1} = 0,$$
$$\lambda_{32} = A_{ij}f_{i3}f_{j2} = 0. \tag{8}$$

Conditions (8) are necessary and sufficient for f_{ip} to be eigenvectors, while expressions (7) give the eigenvalues of matrix A.

As was shown in [3], the matrices $A^{(1)}$, $A^{(2)}$, and $A^{(3)}$ correspond to the characteristic (non-normalized) vectors

$$[\alpha_{j1}, \beta_{j1}, \varepsilon_{jmn}\alpha_{m1}\beta_{n1}],$$
$$[\alpha_{j2}, \beta_{j2}, \varepsilon_{jmn}\alpha_{m2}\beta_{n2}], \quad [\alpha_{j3}, \beta_{j3}, \varepsilon_{jmn}\alpha_{m3}\beta_{n3}]. \tag{9}$$

Taking Eqs. (3) and (7)-(9) into account, we find the eigenvalues of $A^{(1)}$, $A^{(2)}$, and $A^{(3)}$ and the conditions under which expressions (9) are eigenvectors of these matrices. We thus obtain

$$a_{11}^{(1)} = \frac{A_{i(11)j}\alpha_{i1}\alpha_{j1}}{\alpha_{s1}\alpha_{s1}}, \quad a_{11}^{(2)} = \frac{A_{i(11)j}\beta_{i1}\beta_{j1}}{\beta_{s1}\beta_{s1}},$$
$$a_{11}^{(3)} = \frac{A_{i(11)j}\varepsilon_{imn}\alpha_{m1}\beta_{n1}\varepsilon_{jpq}\alpha_{p1}\beta_{q1}}{\alpha_{s1}\alpha_{s1}\beta_{r1}\beta_{r1}}; \tag{10}$$

$$\lambda_{21}^{(1)} = A_{i(11)j}\alpha_{i1}\beta_{j1} = 0, \lambda_{31}^{(1)} = A_{i(11)j}\alpha_{i1}\varepsilon_{jmn}\alpha_{m1}\beta_{n1} = 0,$$
$$\lambda_{32}^{(1)} = A_{i(11)j}\beta_{i1}\varepsilon_{jmn}\alpha_{m1}\beta_{n1} = 0 \tag{11}$$

(here and below, we omit the normalizing factors);

$$a_{22}^{(1)} = \frac{A_{i(22)j}\alpha_{i2}\alpha_{j2}}{\alpha_{s2}\alpha_{s2}},\ a_{22}^{(2)} = \frac{A_{i(22)j}\beta_{i2}\beta_{j2}}{\beta_{s2}\beta_{s2}},$$
$$a_{22}^{(3)} = \frac{A_{i(22)j}\varepsilon_{imn}\alpha_{m2}\beta_{n2}\varepsilon_{jpq}\alpha_{p2}\beta_{q2}}{\alpha_{s2}\alpha_{s2}\beta_{r2}\beta_{r2}}; \tag{12}$$

$$\lambda_{21}^{(2)} = A_{i(22)j}\alpha_{i2}\beta_{j2} = 0,\ \lambda_{31}^{(2)} = A_{i(22)j}\alpha_{i2}\varepsilon_{jmn}\alpha_{m2}\beta_{n2} = 0,$$
$$\lambda_{32}^{(2)} = A_{i(22)j}\beta_{i2}\varepsilon_{jmn}\alpha_{m2}\beta_{n2} = 0; \tag{13}$$

$$a_{33}^{(1)} = \frac{A_{i(33)j}\alpha_{i3}\alpha_{j3}}{\alpha_{s3}\alpha_{s3}},\ a_{33}^{(2)} = \frac{A_{i(33)j}\beta_{i3}\beta_{j3}}{\beta_{s3}\beta_{s3}},$$
$$a_{33}^{(3)} = \frac{A_{i(33)j}\varepsilon_{imn}\alpha_{m3}\beta_{n3}\varepsilon_{jpq}\alpha_{p3}\beta_{q3}}{\alpha_{s3}\alpha_{s3}\beta_{r3}\beta_{r3}}; \tag{14}$$

$$\lambda_{21}^{(3)} = A_{i(33)j}\alpha_{i3}\beta_{j3} = 0,\ \lambda_{31}^{(3)} = A_{i(33)j}\alpha_{i3}\varepsilon_{jmn}\alpha_{m3}\beta_{n3} = 0,$$
$$\lambda_{32}^{(3)} = A_{i(33)j}\beta_{i3}\varepsilon_{jmn}\alpha_{m3}\beta_{n3} = 0. \tag{15}$$

Expressions (9) will be eigenvectors for $A^{(1)}$, $A^{(2)}$, and $A^{(3)}$ if conditions (11), (13), and (15) are satisfied. The eigenvalues are then given by Eqs. (10), (12), and (14).

Since the matrices $A_{i(kl)j}$ are symmetric, then if we multiply Eqs. (3) by the corresponding columns of α_{is}, β_{ip} and take the first three equations into account, we can use (3) to obtain

$$\begin{aligned}
a_{23}^{(1)} &= \frac{A_{i(23)j}\alpha_{i2}\alpha_{j2}}{\alpha_{m2}\alpha_{m2}} = \frac{A_{i(23)j}\alpha_{i3}\alpha_{j3}}{\alpha_{n3}\alpha_{n3}},\\
a_{13}^{(1)} &= \frac{A_{i(13)j}\alpha_{i1}\alpha_{j1}}{\alpha_{m1}\alpha_{m1}} = \frac{A_{i(13)j}\alpha_{i3}\alpha_{j3}}{\alpha_{n3}\alpha_{n3}},\\
a_{12}^{(1)} &= \frac{A_{i(12)j}\alpha_{i1}\alpha_{j1}}{\alpha_{m1}\alpha_{m1}} = \frac{A_{i(12)j}\alpha_{i2}\alpha_{j2}}{\alpha_{n2}\alpha_{n2}};\\
a_{23}^{(2)} &= \frac{A_{i(23)j}\beta_{i2}\beta_{j2}}{\beta_{m2}\beta_{m2}} = \frac{A_{i(23)j}\beta_{i3}\beta_{j3}}{\beta_{n3}\beta_{n3}},\\
a_{13}^{(2)} &= \frac{A_{i(13)j}\beta_{i1}\beta_{j1}}{\beta_{m1}\beta_{m1}} = \frac{A_{i(13)j}\beta_{i3}\beta_{j3}}{\beta_{n3}\beta_{n3}},\\
a_{12}^{(2)} &= \frac{A_{i(12)j}\beta_{i1}\beta_{j1}}{\beta_{m1}\beta_{m1}} = \frac{A_{i(12)j}\beta_{i2}\beta_{j2}}{\beta_{n2}\beta_{n2}}.
\end{aligned} \tag{16}$$

It is evident from (16) that the following conditions must be satisfied

$$\begin{aligned}
A_{i(23)j}\left(\frac{\alpha_{i2}\alpha_{j2}}{\alpha_{m2}\alpha_{m2}} - \frac{\alpha_{i3}\alpha_{j3}}{\alpha_{n3}\alpha_{n3}}\right) &= 0,\\
A_{i(13)j}\left(\frac{\alpha_{i1}\alpha_{j1}}{\alpha_{m1}\alpha_{m1}} - \frac{\alpha_{i3}\alpha_{j3}}{\alpha_{n3}\alpha_{n3}}\right) &= 0,\\
A_{i(12)j}\left(\frac{\alpha_{i1}\alpha_{j1}}{\alpha_{m1}\alpha_{m1}} - \frac{\alpha_{i2}\alpha_{j2}}{\alpha_{n2}\alpha_{n2}}\right) &= 0;\\
A_{i(23)j}\left(\frac{\beta_{i2}\beta_{j2}}{\beta_{m2}\beta_{m2}} - \frac{\beta_{i3}\beta_{j3}}{\beta_{n3}\beta_{n3}}\right) &= 0,\\
A_{i(13)j}\left(\frac{\beta_{i1}\beta_{j1}}{\beta_{m1}\beta_{m1}} - \frac{\beta_{i3}\beta_{j3}}{\beta_{n3}\beta_{n3}}\right) &= 0,\\
A_{i(12)j}\left(\frac{\beta_{i1}\beta_{j1}}{\beta_{m1}\beta_{m1}} - \frac{\beta_{i2}\beta_{j2}}{\beta_{n2}\beta_{n2}}\right) &= 0.
\end{aligned} \tag{17}$$

In (4), the third column has the form $t_{j3} = \gamma_{j(sp)}\partial_{sp}$, where:

$$\gamma_{j(sp)} = \frac{1}{2}\varepsilon_{jmn}(\alpha_{ms}\,\beta_{np} + \alpha_{mp}\,\beta_{ns}). \tag{18}$$

We now write the relation analogous to (2):

$$(A_{i(kl)j} - \delta_{ij}a^{(3)}_{kl})\gamma_{j(sp)}\partial_{klsp} = 0. \tag{19}$$

Reducing similar terms in (19) and equating the coefficients with ∂_{klsp} to zero, we obtain the following system of equations from (19)

$$\begin{gathered}
(A_{i(11)j} - \delta_{ij}a_{11})\gamma_{j(11)} = 0,\ (A_{i(22)j} - \delta_{ij}a_{22})\gamma_{j(22)} = 0,\\
(A_{i(33)j} - \delta_{ij}a_{33})\gamma_{j(33)} = 0;\\
2(A_{i(23)j} - \delta_{ij}a_{23})\gamma_{j(22)} + 2(A_{i(22)j} - \delta_{ij}a_{22})\gamma_{j(23)} = 0,\\
2(A_{i(23)j} - \delta_{ij}a_{23})\gamma_{j(33)} + 2(A_{i(33)j} - \delta_{ij}a_{33})\gamma_{j(23)} = 0;\\
2(A_{i(13)j} - \delta_{ij}a_{13})\gamma_{j(11)} + 2(A_{i(11)j} - \delta_{ij}a_{11})\gamma_{j(13)} = 0,\\
2(A_{i(13)j} - \delta_{ij}a_{13})\gamma_{j(33)} + 2(A_{i(33)j} - \delta_{ij}a_{33})\gamma_{j(13)} = 0;\\
2(A_{i(12)j} - \delta_{ij}a_{12})\gamma_{j(11)} + 2(A_{i(11)j} - \delta_{ij}a_{11})\gamma_{j(12)} = 0,\\
2(A_{i(12)j} - \delta_{ij}a_{12})\gamma_{j(22)} + 2(A_{i(22)j} - \delta_{ij}a_{22})\gamma_{j(12)} = 0;\\
(A_{i(11)j} - \delta_{ij}a_{11})\gamma_{j(22)} + (A_{i(22)j} - \delta_{ij}a_{22})\gamma_{j(11)} + 4(A_{i(12)j} - \delta_{ij}a_{12})\gamma_{j(12)} = 0,\\
(A_{i(11)j} - \delta_{ij}a_{11})\gamma_{j(33)} + (A_{i(33)j} - \delta_{ij}a_{33})\gamma_{j(11)} + 4(A_{i(13)j} - \delta_{ij}a_{13})\gamma_{j(13)} = 0,\\
(A_{i(22)j} - \delta_{ij}a_{22})\gamma_{j(33)} + (A_{i(33)j} - \delta_{ij}a_{33})\gamma_{j(22)} + 4(A_{i(23)j} - \delta_{ij}a_{23})\gamma_{j(23)} = 0;\\
2(A_{i(11)j} - \delta_{ij}a_{11})\gamma_{j(23)} + 2(A_{i(23)j} - \delta_{ij}a_{23})\gamma_{j(11)} +\\
+ 4(A_{i(13)j} - \delta_{ij}a_{13})\gamma_{j(12)} + 4(A_{i(12)j} - \delta_{ij}a_{12})\gamma_{j(13)} = 0,\\
2(A_{i(22)j} - \delta_{ij}a_{22})\gamma_{j(13)} + 2(A_{i(13)j} - \delta_{ij}a_{13})\gamma_{j(22)} +\\
+ 4(A_{i(23)j} - \delta_{ij}a_{23})\gamma_{j(12)} + 4(A_{i(12)j} - \delta_{ij}a_{12})\gamma_{j(23)} = 0,\\
2(A_{i(33)j} - \delta_{ij}a_{33})\gamma_{j(12)} + 2(A_{i(12)j} - \delta_{ij}a_{12})\gamma_{j(33)} +\\
+ 4(A_{i(23)j} - \delta_{ij}a_{23})\gamma_{j(13)} + 4(A_{i(13)j} - \delta_{ij}a_{13})\gamma_{j(23)} = 0.
\end{gathered} \tag{20}$$

Here, $a_{kl} = a_{kl}^{(3)} = a_{lk}^{(3)}$. Taking (18) into account, we find that the first three equations of (20) lead to Eqs. (10), (12), (14) for $a_{11}^{(3)}$, $a_{22}^{(3)}$, $a_{33}^{(3)}$ and conditions (11), (13), (15). Analogously to (16), we find from the other equations of (20) that

$$\begin{aligned}
a^{(3)}_{23} &= \frac{A_{i(23)j}\gamma_{i(22)}\gamma_{j(22)}}{\gamma_{m(22)}\gamma_{m(22)}} = \frac{A_{i(23)j}\gamma_{i(33)}\gamma_{j(33)}}{\gamma_{n(33)}\gamma_{n(33)}},\\
a^{(3)}_{13} &= \frac{A_{i(13)j}\gamma_{i(11)}\gamma_{j(11)}}{\gamma_{m(11)}\gamma_{m(11)}} = \frac{A_{i(13)j}\gamma_{i(33)}\gamma_{j(33)}}{\gamma_{n(33)}\gamma_{n(33)}},\\
a^{(3)}_{12} &= \frac{A_{i(12)j}\gamma_{i(11)}\gamma_{j(11)}}{\gamma_{m(11)}\gamma_{m(11)}} = \frac{A_{i(12)j}\gamma_{i(22)}\gamma_{j(22)}}{\gamma_{n(22)}\gamma_{n(22)}}.
\end{aligned} \tag{21}$$

It is apparent from (21) that the following conditions must be satisfied

$$\begin{aligned}
A_{i(23)j}\left(\frac{\gamma_{i(22)}\gamma_{j(22)}}{\gamma_{m(22)}\gamma_{m(22)}} - \frac{\gamma_{i(33)}\gamma_{j(33)}}{\gamma_{n(33)}\gamma_{n(33)}}\right) &= 0,\\
A_{i(13)j}\left(\frac{\gamma_{i(11)}\gamma_{j(11)}}{\gamma_{m(11)}\gamma_{m(11)}} - \frac{\gamma_{i(33)}\gamma_{j(33)}}{\gamma_{n(33)}\gamma_{n(33)}}\right) &= 0,\\
A_{i(12)j}\left(\frac{\gamma_{i(11)}\gamma_{j(11)}}{\gamma_{m(11)}\gamma_{m(11)}} - \frac{\gamma_{i(22)}\gamma_{j(22)}}{\gamma_{n(22)}\gamma_{n(22)}}\right) &= 0.
\end{aligned} \tag{22}$$

Thus, the algorithm for constructing operators T and D will be as follows. We specify the matrices α_{js}, β_{jp}, which are connected by condition (5). We then require satisfaction of necessary conditions (11), (13), (15), (17), (22), which we use to find possible values of the matrices $A_{i(kl)j}$. We then use Eqs. (10), (12), (14), (16), and (21) to obtain the coefficients

$a_{kl}^{(1)}$, $a_{kl}^{(2)}$, $a_{kl}^{(3)}$. Finally, we check to make sure that all remaining equations of (3) and (20) are satisfied. We used this method and obtained basically new solutions.

The operators T and D were found earlier [2, 3] for isotropic and transversely isotropic materials. Now we construct a general definition of certain classes of anisotropic materials with arbitrary parameters using the operators T and D. To illustrate, if $\alpha_{js} = \partial_{js}$, then by following the algorithm and performing all necessary calculations we obtain a generalization of a Green medium [9, 10]:

$$A_{ij} = \begin{bmatrix} A_{11} & & & & & \\ A_{11} - A_{66} & A_{11} & & & \text{sym} & \\ A_{11} - A_{55} & A_{11} - A_{44} & A_{11} & & & \\ A_{41} & 0 & 0 & A_{44} & & \\ 0 & A_{52} & 0 & \frac{-A_{63}}{\sqrt{2}} & A_{55} & \\ 0 & 0 & A_{63} & \frac{-A_{52}}{\sqrt{2}} & \frac{-A_{41}}{\sqrt{2}} & A_{66} \end{bmatrix}. \tag{23}$$

Here

$$A_{41} = -2\sqrt{2}bc_2c_3;\ A_{44} = 2(a - bc_1^2);$$
$$A_{52} = -2\sqrt{2}bc_1c_3;\ A_{55} = 2(a - bc_2^2);$$
$$A_{63} = -2\sqrt{2}bc_1c_2;\ A_{66} = 2(a - bc_3^2),$$

while the operators T and D have the form

$$T = [\partial_j,\ \epsilon_{jmn}c_m\partial_n,\ c_j\partial_{kk} - c_m\partial_{mj}]; \tag{24}$$

$$\begin{aligned} D_1 &= A_{11}\partial_{kk} - \rho\partial_{..}, \\ D_2 &= [(a - bc_mc_m)\delta_{kl} + bc_kc_l]\partial_{kl} - \rho\partial_{..}, \\ D_3 &= a\partial_{kk} - \rho\partial_{..}, \end{aligned} \tag{25}$$

where A_{11}, a, b, c_1, c_2, and c_3 are arbitrary parameters such that matrix (23) is positive definite [11, 12]. When b = 0, Eq. (23) yields an isotropic material.

If we replace ∂_k by n_k and replace $\partial_{..}$ by $|v|^2 = v_iv_i$ ($v_i = |v| n_i$), we can use (25) to find the phase velocities corresponding to the wave normal n_k:

$$\rho|v|_1^2 = A_{11}n_kn_k = A_{11},$$
$$\rho|v|_2^2 = [(a - bc_mc_m)\delta_{kl} + bc_kc_l]n_kn_l = a - bc_mc_m + bc_kc_ln_kn_l,$$
$$\rho|v|_3^2 = an_kn_k = a.$$

It is evident from (24) that Eqs. (23)-(25) define a medium with purely longitudinal and purely transverse waves [13], regardless of the direction of the wave normal n_k. Since $|L| = |D| = D_1D_2D_3$, it follows from (25) that the equations of the phase velocity surface and refraction surface [13] decompose into independent equations of three surfaces. These properties are essentially new properties for anisotropic materials, apart from the cases of isotropic and transversely isotropic materials [13, p. 165].

We now use (23) to find the strain energy:

$$2\Phi = A_{ij}\varepsilon_i\varepsilon_j = A_{11}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2 + A_{44}(\varepsilon_4^2 - 2\varepsilon_3\varepsilon_2) + A_{55}(\varepsilon_5^2 - 2\varepsilon_3\varepsilon_1) + A_{66}(\varepsilon_6^2 - 2\varepsilon_2\varepsilon_1)$$
$$+ A_{41}(2\varepsilon_4\varepsilon_1 - \sqrt{2}\varepsilon_6\varepsilon_5) + A_{52}(2\varepsilon_5\varepsilon_2 - \sqrt{2}\varepsilon_6\varepsilon_4) + A_{63}(2\varepsilon_6\varepsilon_3 - \sqrt{2}\varepsilon_5\varepsilon_4) \tag{26}$$

($\varepsilon_1 = \varepsilon_{11}, \ldots, \varepsilon_6 = \sqrt{2}\varepsilon_{21}$ represents strains). An expression that is similar to (26) but does not have the last three terms was presented in [10, p. 313] for 2Φ for a Green medium. The author of [10] also wrote that Green's expression "is the most general form of the function Φ for which the propagation of transverse plane waves is possible, i.e. for which the displacements will be parallel to the wave front." Equations (23) and (26) show that the expression in [10, p. 313] is not the most general form for 2Φ, since it has no terms with A_{41}, A_{52}, A_{63} — which may also be nontrivial.

If T has the form ($\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq 0$)

$$T = \begin{bmatrix} \alpha_1\partial_1 & c_2\alpha_3\partial_3 - c_3\alpha_2\partial_2 & c_1t_{p1}t_{p1} - c_nt_{n1}\alpha_1\partial_1 \\ \alpha_2\partial_2 & c_3\alpha_1\partial_1 - c_1\alpha_3\partial_3 & c_2t_{p1}t_{p1} - c_nt_{n1}\alpha_2\partial_2 \\ \alpha_3\partial_3 & c_1\alpha_2\partial_2 - c_2\alpha_1\partial_1 & c_3t_{p1}t_{p1} - c_nt_{n1}\alpha_3\partial_3 \end{bmatrix},$$
$$t_{p1}t_{p1} = \alpha_1^2\partial_{11} + \alpha_2^2\partial_{22} + \alpha_3^2\partial_{33},$$
$$c_nt_{n1} = c_1\alpha_1\partial_1 + c_2\alpha_2\partial_2 + c_3\alpha_3\partial_3,$$

then the Hooke's law matrix

$$A_{ij} = \begin{bmatrix} a + b\alpha_1^2 & & & & & \\ b\alpha_1\alpha_2 - a & a + b\alpha_2^2 & & \text{sym} & & \\ b\alpha_1\alpha_3 - a & b\alpha_2\alpha_3 - a & a + b\alpha_3^2 & & & \\ 0 & 0 & 0 & 2a & & \\ 0 & 0 & 0 & 0 & 2a & \\ 0 & 0 & 0 & 0 & 0 & 2a \end{bmatrix}, \tag{27}$$

Here, the characteristic operators

$$D_1 = (a + b\alpha_1^2)\partial_{11} + (a + b\alpha_2^2)\partial_{22} + (a + b\alpha_3^2)\partial_{33} - \rho\partial \;,$$
$$D_2 = D_3 = a\partial_{kk} - \rho\partial \;. \tag{28}$$

In this equation, a, b, α_i, and c_i are arbitrary parameters. The conditions of positive-definiteness of matrix (27) are as follows: $a > 0$, $b > 0$, $b(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) > \alpha$. Material (27) is a special case of an orthotropic material.

Let

$$T = \begin{bmatrix} \partial_1 & -\partial_2 & -\alpha\partial_{13} \\ \partial_2 & \partial_1 & -\alpha\partial_{23} \\ \alpha\partial_3 & 0 & \partial_{11} + \partial_{22} \end{bmatrix}, \quad \begin{matrix} \alpha_1 = \alpha_2 = 1,\ \alpha_3 = \alpha, \\ c_1 = c_2 = 0,\ c_3 = 1, \end{matrix}$$

then [3] ($\alpha^2 \neq 1$)

$$A_{ij} = \begin{bmatrix} A_{11} & & & & & \\ A_{21} & A_{11} & \text{sym} & & & \\ \dfrac{A_{11}\alpha - A_{33}}{1 - \alpha} & A_{31} & A_{33} & & & \\ 0 & 0 & 0 & \dfrac{2(A_{33} - A_{11}\alpha^2)}{1 - \alpha^2} & & \\ 0 & 0 & 0 & 0 & A_{44} & \\ 0 & 0 & 0 & 0 & 0 & A_{11} - A_{21} \end{bmatrix}; \tag{29}$$

$$D_1 = A_{11}(\partial_{11} + \partial_{22}) + A_{33}\partial_{33} - \rho\partial \ ,$$

$$D_2 = \frac{1}{2}(A_{11} - A_{21})(\partial_{11} + \partial_{22}) + \frac{1}{2}A_{44}\partial_{33} - \rho\partial \ ,$$

$$D_3 = \frac{1}{2}A_{44}\partial_{kk} - \rho\partial \ . \tag{30}$$

Equations (29) define the subclass of transversely isotropic media. There are actual materials that conform to our relations; for example, for hexagonal crystals of cadmium [13], the parameter $\alpha \approx 0.575$.

If for example:

$$T = \begin{bmatrix} \partial_1 & -\partial_2 & \partial_{13} \\ \partial_2 & \partial_1 & \partial_{23} \\ -\partial_3 & 0 & \partial_{11} + \partial_{22} \end{bmatrix},$$

then

$$A_{ij} = \begin{bmatrix} A_{11} & & & & & \\ A_{21} & A_{11} & & \text{sym} & & \\ -A_{11} & -A_{11} & A_{11} & & & \\ 0 & 0 & 0 & A_{44} & & \\ 0 & 0 & 0 & 0 & A_{44} & \\ 0 & 0 & 0 & 0 & 0 & A_{11} - A_{21} \end{bmatrix}; \tag{31}$$

$$D_1 = A_{11}\partial_{kk} - \rho\partial \ ,$$

$$D_2 = \frac{1}{2}(A_{11} - A_{21})(\partial_{11} + \partial_{22}) + \frac{1}{2}A_{44}\partial_{33} - \rho\partial \ ,$$

$$D_3 = \frac{1}{2}A_{44}\partial_{kk} - \rho\partial \ .$$

Matrix (31) cannot be made positive definite. It is therefore physically impossible for a material with such elastic moduli to exist, even though all of the remaining conditions and equations are satisfied.

The examples presented above do not exhaust all of the cases in which Eqs. (1) reduce to three independent wave equations $D\varphi = f$. The other variants cannot be written out. As was noted above, Eqs. (28) and (30) can be used to find the phase velocities corresponding to the wave normal n_k, while the equations of the wave surfaces [13] decompose into independent equations of three surfaces.

Note. When we assign α_{js} and find β_{jp} from the relation

$$\alpha_{js}\beta_{jp} = c_{sp} = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix},$$

we must consider all possible values of the parameters c_1, c_2, c_3, rather than just the variant in which all c_i are equal to zero. More general classes of anisotropic materials allowing the operators T and D can be obtained in certain cases for trivial values of c_i.

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